Homework assignment 2

Due date: October 19, 2021

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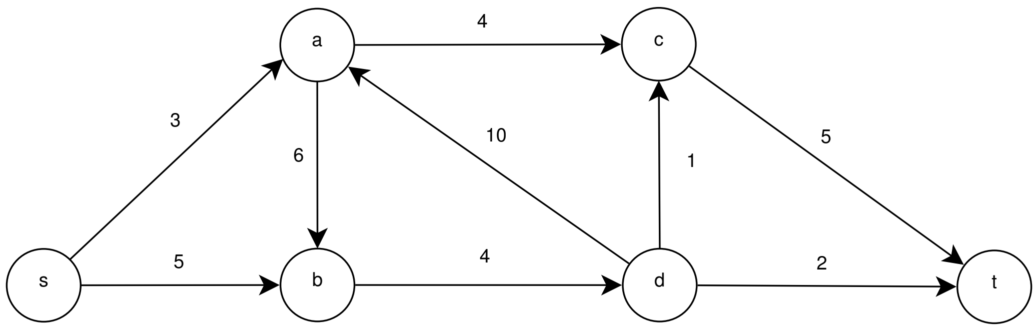
1. **30 points.** Find a maximum flow between s and t in the following network by using Edmonds-Karp algorithm:



Demonstrate the main steps in the algorithm. Show all minimum cuts. How many different minimum cuts can you find?

2. **35 points.** You are given a maximum *integer* flow function $f : \mathcal{E} \to \mathbb{N} \cup \{0\}$ in the flow network $\mathcal{N}(\mathcal{G}(\mathcal{V}, \mathcal{E}), s, t, c)$, where $\mathcal{G}$ is a finite directed graph, and $c : \mathcal{E} \to \mathbb{N}$ is an *integer* capacity function. The capacities of the seven edges $e_1, e_2, \cdots, e_7$ are increased by 1 each. It is required to find a maximum flow in the new network. Describe an algorithm with complexity $O(|\mathcal{E}|)$ that does that, and prove its correctness.

3. **35 points.** The group of n students participates in the sports competition. There are four sports teams: the basketball team that has m_1 vacancies, the soccer team that has m_2 vacancies, the cycling team that has m_3 vacancies, and the running team that has m_4 vacancies. Denote $m = m_1 + m_2 + m_3 + m_4$. Each student chooses a subset of teams that he/she is capable to participate in.

   Propose an algorithm that assigns the students to the teams, under the following conditions:

   - Each student participates in at most one team.
   - The algorithm can only assign the student to the team that he/she has selected.
   - The number of students that are assigned to each team is equal to the number of vacancies.

   If there is no suitable assignment of students to the teams, the algorithm will output an appropriate message.

   Prove the correctness of the proposed algorithm. The required time complexity is $O(n \cdot m)$.

4. **Bonus: 20 points.** Let $\mathcal{N}(\mathcal{G}(\mathcal{V}, \mathcal{E}), s, t, c)$ be a flow network. Consider two sets of vertices S_1 and S_2, where

$$\{s\} \subseteq S_1 \subseteq \mathcal{V} \setminus \{t\} \text{ and } \{s\} \subseteq S_2 \subseteq \mathcal{V} \setminus \{t\} \ .$$

It is known that both $(S_1 : \overline{S}_1)$ and $(S_2 : \overline{S}_2)$ are minimum cuts in $\mathcal{N}$. Define sets of vertices $T_1 = S_1 \cup S_2$ and $T_2 = S_1 \cap S_2$. Prove that both $(T_1 : \overline{T}_1)$ and $(T_2 : \overline{T}_2)$ are minimum cuts too.